Inhomogeneous Continuity Equation with Application to Hamiltonian ODE

(joint work with L. Chayes & W. Gangbo)

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Preliminaries

Motivation

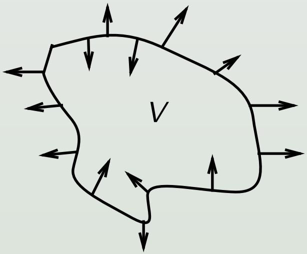
Inhomogeneous Continuity Equation

Continuity Equation I

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0$$

ρ = (probability) density

v = velocity field

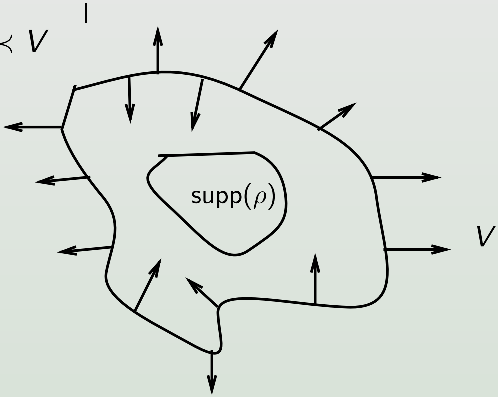


$\triangle$ in mass = flux in/out of volume:

$$\frac{dM_V}{dt} = \int \frac{\partial \rho}{\partial t}\, dt = -\int_V \nabla \cdot (\rho v)\, dx = -\int_{\partial V} \rho\; v \cdot \hat{n}\; dS$$

Continuity Equation II

$\frac{dM_V}{dt} = 0$ if the supp$(\rho) \prec V$



For measures:

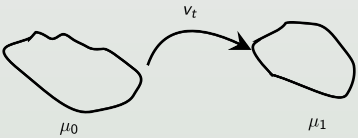
$$\partial_t \mu_t + \nabla \cdot (\mu_t v_t) = 0$$

in the weak sense:

$$\int_0^T \int \partial_t \varphi + \langle v_t, \nabla \varphi \rangle \; d\mu_t \; dt = 0 \qquad \forall \varphi \in C_c^\infty(\mathbb{R}^d \times (0,T))$$

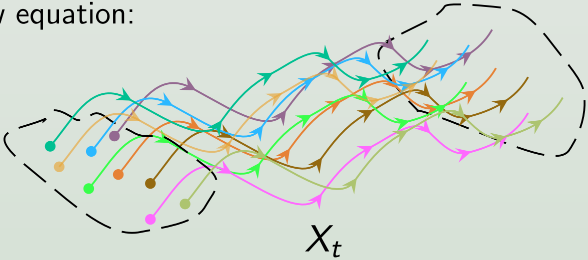
Lagrangian Description I

Eulerian:



Given v_t, have flow equation:

$$\begin{cases} \dot{X}_t = v_t(X_t) \\ X_0 = \mathrm{id} \end{cases}$$



Lagrangian Description II

Define

$$\mu_t = X_t \# \mu_0$$

(Here $T\#\mu = \nu$ if for any measurable A

$$\nu(A) = \mu(T^{-1}(A))$$

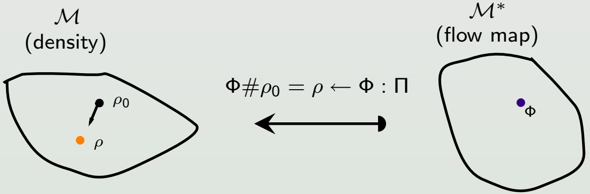
or for any test function $\varphi \in L^1(d\nu)$

$$\int \varphi(y)\, d\nu(y) = \int \varphi(T(x))\, d\mu(x)$$)

Then (formally), $\{\mu_t\}_{t\in[0,T]}$ satisfy the continuity equation:

$$\varphi \in C_c^\infty(\mathbb{R}^d \times (0,T)); \quad \Psi(x,t) = \varphi(X_t(x), t)$$

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^d} \partial_t \varphi(x) + \langle v_t(x), \varphi(x) \rangle \; d\mu_t(x)\; dt \\
&= \int_0^T \int_{\mathbb{R}^d} \partial_t \varphi(X_t(x), t) + \langle v_t(X_t(x), \nabla \varphi(X_t(x)) \rangle \; d\mu_0(x)\; dt \\
&= \int_0^T \int_{\mathbb{R}^d} \frac{d\Psi}{dt}(x,t)\; d\mu_0(x)\; dt \\
&= \int_{\mathbb{R}^d} \varphi(X_T(x), T) - \varphi(x, 0)\; d\mu_0(x) \\
&= 0
\end{aligned}$$



$s = -\nabla \cdot (\rho \nabla p)$; $\quad g_\rho(s_1, s_2) = \int \rho \nabla \rho_1 \cdot \rho_2$
(non-flat)

$g^*_\Phi(v_1, v_2) = \int (v_1 \cdot v_2)\rho_0$
(flat)

(Induced distance:

$d(x_0, x_1)^2 = \inf\{\int_0^1 g_{x(t)}(\frac{dx}{dt}, \frac{dx}{dt})\, dt : t \mapsto x(t) \in \mathcal{M}', x(0) = x_0, x(1) = x_1\}$)

Upshot:

$$d(\rho_0, \rho)^2 = \inf_{\Phi:\rho=\Phi\#\rho_0} \int \rho_0 |\mathrm{id} - \Phi|^2$$

F. Otto.
The geometry of dissipative evolution eqns: the porous medium equation.
Comm. PDE, 26 (2001), 101-174.

A.C. Curves and the Continuity Equation

Definition. Let

$$\mathscr{P}_2(\mathbb{R}^d, W_2)$$

denote the space of probability measures with bounded second moment equipped with the Wasserstein distance

$$W_2^2(\mu,\nu) = \min\left\{\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2\, d\gamma(x,y) : \gamma\in\Gamma(\mu,\nu)\right\}$$

and

$$\Gamma(\mu,\nu) = \{\gamma : \gamma(A\times\mathbb{R}^d) = \mu(A) \text{ and } \gamma(\mathbb{R}^d\times B) = \nu(B), \text{ for all measurable } A \text{ and } B\}$$

Theorem. There is a correspondence:

$$\{\text{A.C. curves in } \mathscr{P}_2(\mathbb{R}^d, W_2)\} \Longleftrightarrow \{\text{velocity fields } v_t\in L^2(d\mu_t)\}$$

via

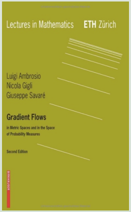
$$\partial_t\mu_t + \nabla\cdot(v_t\mu_t) = 0 \quad \text{and} \quad \lim_{h\to 0}\frac{1}{|h|}W_2(\mu_{t+h},\mu_t)(\leq) = \|v_t\|_{L^2(\mu_t)}$$

Thus

$$W_2^2(\mu_0,\mu_1) = \min\left\{\int_0^1 \|v_t\|^2_{L^2(d\mu_t)} : \partial_t\mu_t + \nabla\cdot(v_t\mu_t) = 0\right\}$$

and

$$T_\mu\mathscr{P}_2(\mathbb{R}^d, W_2) = \overline{\{\nabla\varphi : \varphi\in C_c^\infty(\mathbb{R}^d)\}}^{L^2(d\mu)}$$



Hamiltonian ODE I

Hamiltonian Dynamics. $\mathbb{R}^{2d} \ni x = (p, q) = (\text{momentum, position})$

$$\text{E.g.,} \quad H(p,q) = \frac{1}{2}|p|^2 + \Phi(q)$$

$$\dot{x} = \begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix} \begin{pmatrix} H_p \\ H_q \end{pmatrix} = \mathbb{J}\nabla H$$

Start with measure, infinite dimensional Hamiltonian system?

Definition (Hamiltonian ODE). $\mathscr{H} : \mathscr{P}_2(\mathbb{R}^{2d}) \to (-\infty, \infty]$ (proper, lowersemicontinuous).
A.C. curve $\{\mu_t\}_{[0,T]}$ is *Hamiltonian ODE* w.r.t. $\mathscr{H}$ if

$$\exists v_t \in L^2(d\mu_t), \quad \|v_t\|_{L^2(d\mu_t)} \in L^1(0,T)$$

such that

$$\begin{cases} \partial_t \mu_t + \nabla \cdot (\mathbb{J} v_t \mu_t) = 0, & \mu_0 = \overline{\mu}, \quad t \in (0,T) \\ v_t \in T_{\mu_t}\mathscr{P}_2(\mathbb{R}^{2d}) \cap \partial \mathscr{H}(\mu_t) & \text{for a.e., t} \end{cases}$$

L. Ambrosio and W. Gangbo. *Hamiltonian ODE's in the Wasserstein Space of Probability Measures.* Comm. in Pure and Applied Math., **61**, 18–53 (2007).

W. Gangbo, H. K. Kim, and T. Pacini. *Differential forms on Wasserstein space and infinite dimensional Hamiltonian systems.* To appear in Memoirs of AMS.

Hamiltonian ODE II

Example.

$$\mathscr{H}(\mu) = \frac{1}{2}\int |p|^2 \, d\mu + \int \Phi(q) \, d\mu + \frac{1}{2}\int (W * \mu)(q) \, d\mu$$

$$\nabla\mathscr{H}(\mu) = (p, -(\nabla W * \mu + \Phi)(q))$$

Theorem. (Ambrosio, Gangbo) Suppose $\mathscr{H} : \mathscr{P}_2(\mathbb{R}^{2d}) \to \mathbb{R}$ satisfies

♣$|\nabla\mathscr{H}(x)| \leq C(1+|x|)$

∘ If $\mu_n = \rho_n \mathscr{L}^{2d}, \mu = \rho\mathscr{L}^{2d}$ and $\mu_n \rightharpoonup \mu$ then $\nabla\mathscr{H}(\mu_{n_k})\mu_{n_k} \rightharpoonup \nabla\mathscr{H}(\mu)\mu$

Then given $\overline{\mu} = \overline{\rho}\mathscr{L}^{2d}$:

∘The Hamiltonian ODE admits a solution for $t \in [0, T]$

∘ $t \mapsto \mu_t$ is L–Lipschitz

∘ If $\mathscr{H}$ is λ–convex, then $\mathscr{H}(\mu_t) = \mathscr{H}(\overline{\mu})$.

Mass Reaching Infinity in Finite Time

Condition (♣).

We are solving

$$\partial_t \mu_t + \nabla \cdot (\mathbb{J}\nabla \mathscr{H} \mu_t) = 0; \quad v_t := \mathbb{J}\nabla\mathscr{H}(\mu_t)$$

Recall characteristics

$$\dot{X}_t = v_t(X_t); \quad X_0 = \mathrm{id}$$

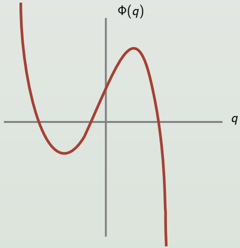
$|v_t(x)| \le C(1+|x|) \Longrightarrow |X_t| \lesssim e^{Ct}(1+|X_0|)$:
preserves compact support, second moment...

Explicit Computation. $|v_t(X_t)| = C(1+|X_t|)^R$, $R > 1$

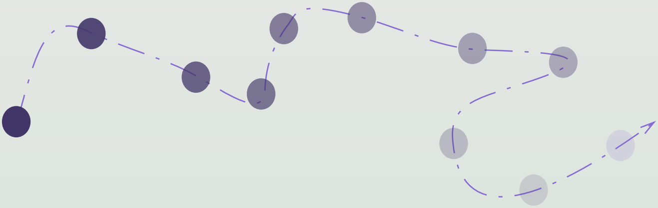
$$\left(\frac{|X_t|}{|X_0|}\right)^{R-1} = \frac{1}{1 - t(R-1)|X_0|^{R-1}}$$

$$x \rightsquigarrow \infty \quad \text{at time} \quad \tau(x) = \frac{1}{(R-1)|x|^{R-1}} < \infty$$

What about other Hamiltonians? E.g.,



Regularization: Fade With Arc Length



$$\dot{X}_t = v_t(X_t) \qquad M_t = M_0 e^{-\int_0^t C_s(X_s)|v_s(X_s)|\ ds}$$

For simplicity, $C_s := \varepsilon$

Inhomogeneous Continuity Equation

$$(\spadesuit) \qquad \frac{\partial \mu_t}{\partial t} + \nabla \cdot (v_t \mu_t) = -\varepsilon |v_t| \mu_t$$

Given μ_0, v_t, define

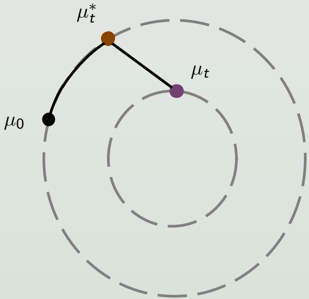
$$\mu_t^* = X_t \# \mu_0$$

$$R_t(X_t) = \exp(-\varepsilon \int_0^t |v_t(X_s) \; ds)$$

then

$$\mu_t = R_t \mu_t^*$$

satisfies ($\spadesuit$).



Proposition. ($\spadesuit$) preserves α–exponential moments for $\alpha \leq \varepsilon$, since

distance tranveled $\leq$ arclength

A Distance for Measures I

Observation. If D_1 and D_2 are distances, then so is $D' = \sqrt{D_1^2 + D_2^2}$.

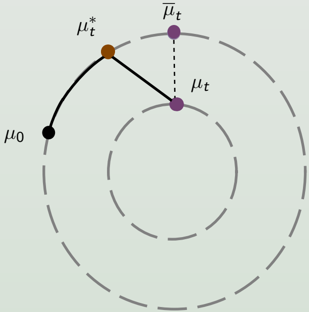
Fix $\varepsilon > 0$ and consider
$\mathscr{M}_{\infty,\varepsilon}(\mathbb{R}^{2d}, B_2)$:

$\{$(positive) Borel measures

with ε–exponential moment$\}$

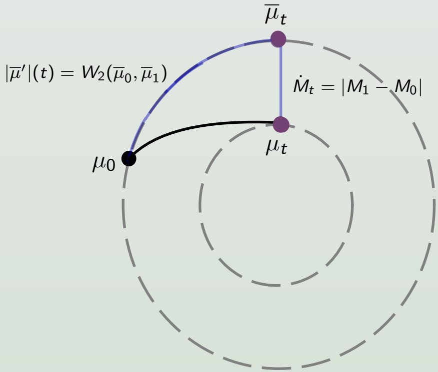
with distance

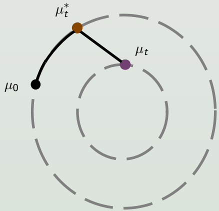
$$B_2^2(\mu,\nu) = W_2^2(\overline{\mu},\overline{\nu}) + (M_\mu - M_\nu)^2$$



A Distance for Measures II

Geodesics of B_2. Geodesic in $(\mathscr{P}_2, W_2)$ + linear decay of mass

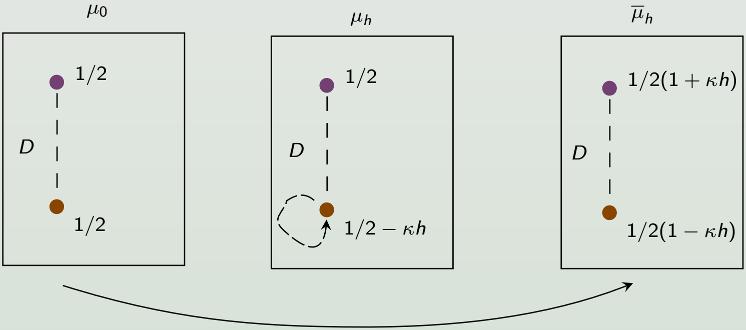




$$|\mu'|(t) = \sqrt{|\overline{\mu}'|(t))^2 + (\dot{M}_t)^2}$$

Continuity Of Dynamics I

Example. Hölder–1/2 Continuity; moment assumption needed.



$$W_2 = D\sqrt{\kappa h}$$

Continuity of Dynamics II

Lemma. Let $\mu_0 \in \mathscr{M}_{\infty,\varepsilon}$. Let us assume that we have (time–dependent) velocity fields v_t satisfying

$$|v_t(x)| \leq C(1+|x|)^R$$

for some constants $C, R > 0$. Then if $(\mu_t^\varepsilon)_{t\in[0,T]}$ is a solution to

$$\frac{\partial \mu_t^\varepsilon}{\partial t} + \nabla \cdot (v_t \mu_t^\varepsilon) = -\varepsilon |v_t| \mu_t^\varepsilon,$$

$\exists(C, R, \varepsilon)$–dependent constant $G < \infty$ such that $\forall t, t+h \in [0, T]$ with $h < h_0$ for some $h_0 > 0$ sufficiently small

$$B_2(\mu_t^\varepsilon, \mu_{t+h}^\varepsilon) \leq G M_{\infty,\varepsilon}(\mu_0)\sqrt{h}$$

Application to Hamiltonian ODE I

Theorem. (Chayes, Gangbo, L.) Fix $\varepsilon > 0$ and $T > 0$.
Suppose $\mathscr{H} : \mathscr{M}_{\infty,\varepsilon} \to \mathbb{R}$ and $v_\mu := \mathbb{J}\nabla\mathscr{H}(\mu)$ satisfies

- $v_\mu(x) \leq C(1 + |x|)^R$
- If $\mu_n \rightharpoonup \mu$ narrowly, then $\mu_n v_{\mu_n} \to \mu v_\mu$

Then given $\mu_0 \in \mathscr{M}_{\infty,\varepsilon}$, there exists a solution to

$$\frac{\partial \mu_t^\varepsilon}{\partial t} + \nabla \cdot (v_t \mu_t^\varepsilon) = -\varepsilon |v_t| \mu_t^\varepsilon, \quad t \in [0, T]$$

with

$$v_t^\varepsilon = \mathbb{J}\nabla\mathscr{H}(\mu_t^\varepsilon).$$

Furthermore, there exists $\varepsilon \to 0$ limiting measures $\{\mu_t\}_{t\in[0,T]}$.

Application to Hamiltonian ODE II

Current Work.

- Appropriate limiting measures satisfy the continuity equation.
- Dependence on limiting procedure.
- Appropriate conservation laws (mass, energy, etc.).

Questions.

- Different inhomogeneous equation?
- Different distance?
- Relation between the two?
- Physical systems of relevance?

Thank you